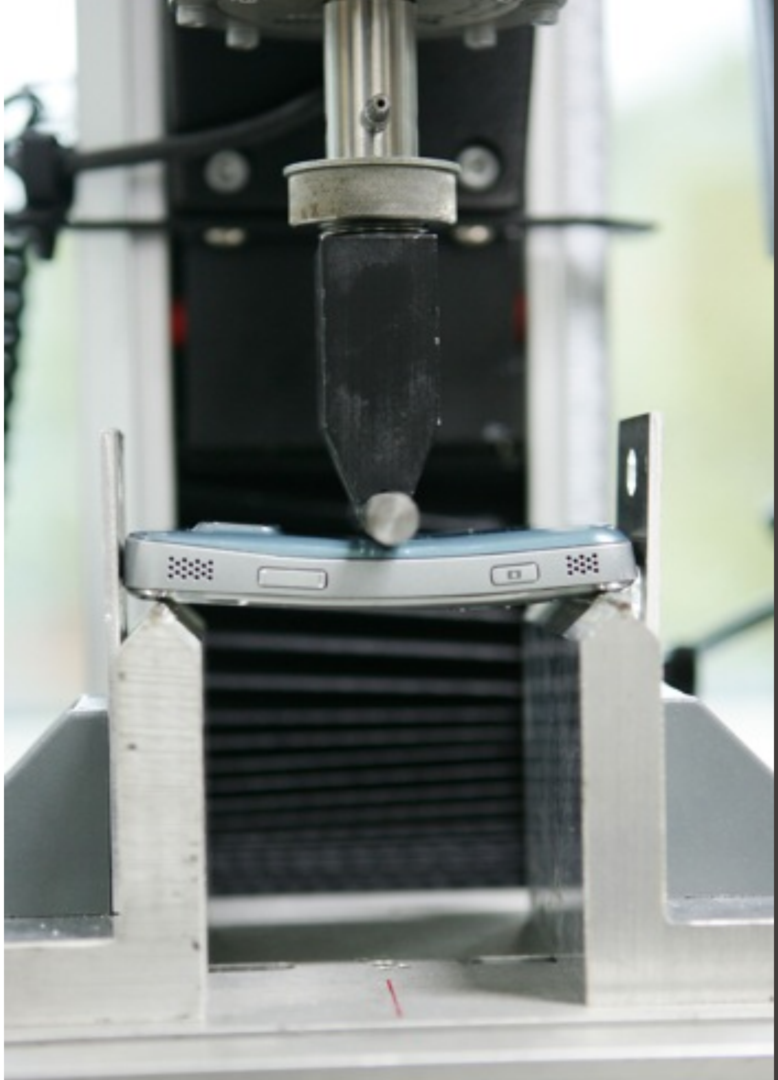
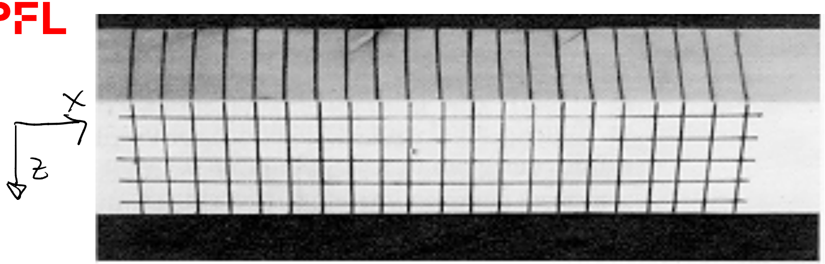


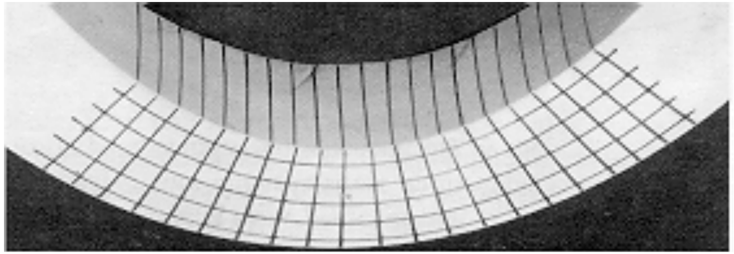
Normal stresses in beams

Georg Fantner

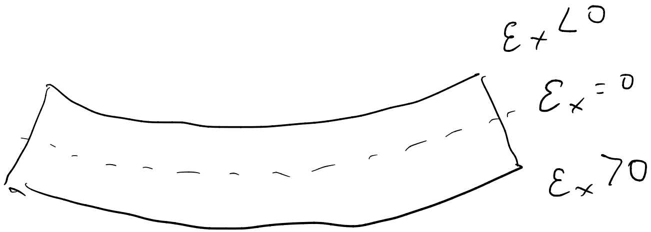
Georg Fantner

Normal Stress in Beams





NEUTRAL AXIS: $\varepsilon_x = 0$



$$\varepsilon_x = \varepsilon_x(z)$$

What normal stress develops due to shear loading?

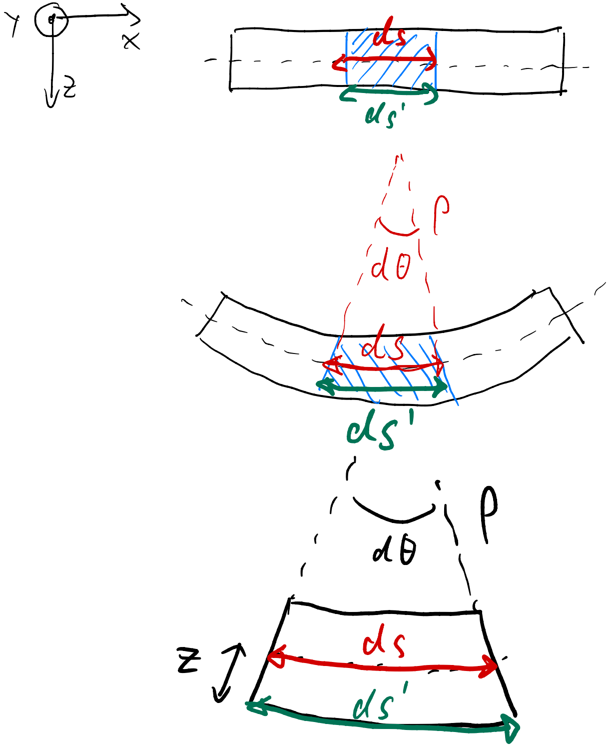
Assume a beam with a symmetric cross-section loaded in pure bending

Our approach: Similar as for torsion, we make assumption about planar cross sections before and after deformation

Planes normal to the beam axis before bending will remain plane after bending

Beam axis: a line parallel to the long axis of the beam that passes through the centroid of the cross section.

ASSUME A SIMPLY SUPPORTED BEAM UNDER LOAD SUCH THAT THE BEAM AXIS IS PART OF A CIRCLE WITH RADIUS ρ



- $ds = \rho \cdot d\theta \Rightarrow \frac{d\theta}{ds} = \frac{1}{\rho} := \kappa$ ... CURVATURE

- LOOK AT ANOTHER CURVE // TO THE BEAM AXIS A DISTANCE z AWAY

$$ds' = (\rho + z)\, d\theta$$

- STRAIN: $\varepsilon_x = \frac{ds' - ds}{ds}$

$$ds' - ds = (\rho + z)\, d\theta - \rho\, d\theta = z\, d\theta$$

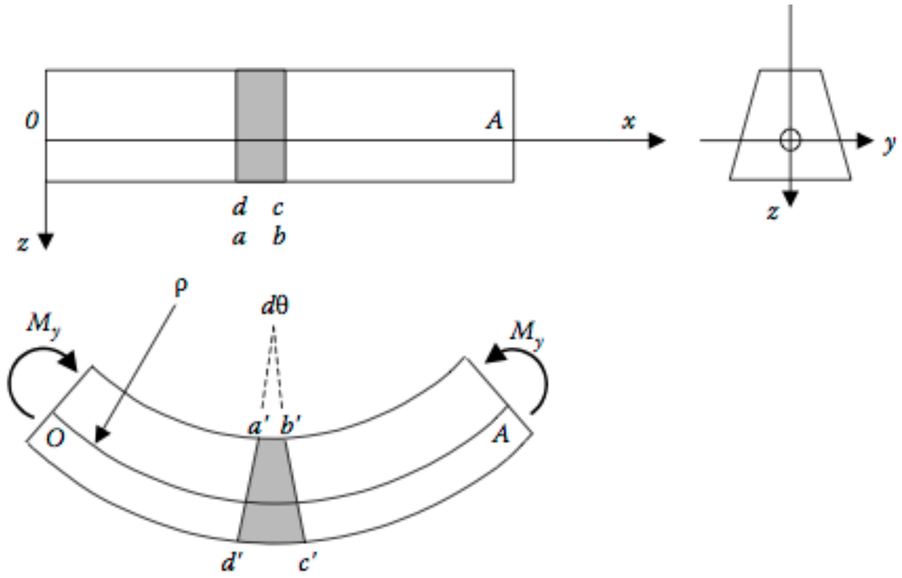
$$\boxed{\varepsilon_x(z) = z \cdot \frac{d\theta}{ds} = \kappa z}$$

$$\varepsilon_x(z) = z \cdot \frac{d\theta}{ds} = \kappa z$$

- FOR $z = 0$ $\Rightarrow$ $\varepsilon_x = 0$ $\Rightarrow$ NEUTRAL AXIS

- HOOKE'S LAW:

$$\sigma_x(z) = E\,\varepsilon_x(z) = E \kappa z$$



Normal Stress in Beams

Normal stress as a function of z

- We want to find an expression for the stress along the beam direction as a function of the position along the beam thickness.

$$\sigma_x = \sigma_x(z)$$

- Assume a bent, simple supported uniform, and symmetric beam. The beam axis then part of a circle with radius ρ.

Normal Stress in Beams

Normal stress as a function of z

- For the curvature κ of the beam we get: $$\kappa = \frac{d\theta}{ds} = \frac{1}{\rho}$$
- For a prismatic beam, ρ and κ are constants.
- Look for another curve parallel to the beam axis at distance z away and calculate the new ds'. In the unbent beam ds'=ds.
- The strain is then:

$$\varepsilon_x = \frac{ds' - ds}{ds} = z\frac{d\theta}{ds} = \kappa \cdot z$$

- This also means that fibers on the beam axis feel no strain

Normal Stress in Beams

Normal stress as a function of z

- We can apply Hooke's law:

$$\varepsilon_x = \kappa \cdot z$$

$$\boxed{\sigma_x(z) = E \cdot \varepsilon_x(z) = E \cdot \kappa \cdot z}$$

- Neutral Axis: axis along the beam that is neither in tension nor in compression. This is at z=0. The neutral axis is the axis through the centroid of the cross section.

Normal stress as a function of z and M

- Next, we want to calculate an expression for the normal stress that relates it to the bending moment M.
- First, we derive an expression for κ as a function of the internal moment and the beam geometry:

$$\kappa = \frac{M_y}{EI_y}$$

- Second moment of area:

$$I_y = \int_A z^2 dA$$

- Using Hooke's law in bending we get:

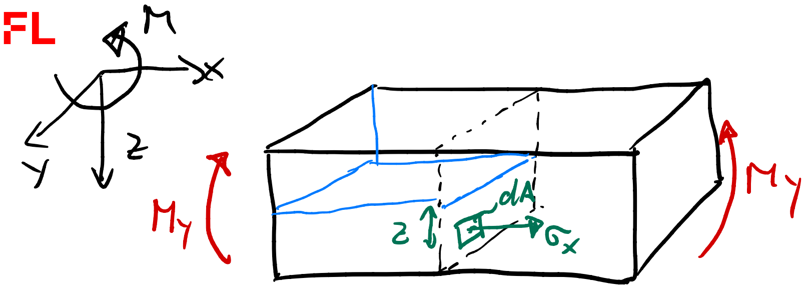
$$\sigma_x(z) = E \cdot \kappa \cdot z = \frac{M_y}{I_y} \cdot z$$

- Since M can vary with x:

$$\boxed{\sigma_x(x,z) = \frac{M_y(x)}{I_y} \cdot z}$$

flexure formula

Georg Fantner



$$dM = \sigma_x \cdot dA \cdot z$$

INTEGRATE OVER AREA A:

$$M_y = \int_A \sigma_x \cdot z \, dA = \int_A E \kappa z \cdot z \, dA = E\kappa \underbrace{\int_A z^2 \, dA}$$

I_y :: SECOND MOMENT OF AREA

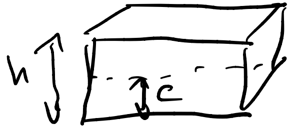
$$\boxed{M_y = E \kappa I_y}$$

$$\boxed{\kappa = \frac{M_y}{E I_y}}$$

ADDING THIS INTO HOOKE'S LAW:

$$\sigma_x(x,z) = \frac{M_y(x)}{I_y} \cdot z$$

FLEXURE FORMULA.



TORSION: $\tau(r) = \frac{T}{J} \cdot r$

- MAXIMUM STRESS IN BENDING: z_{max} IS AT $z = c = \frac{h}{2}$

Normal Stress in Beams

$$\overleftrightarrow{\tau} = \begin{pmatrix} \sigma_x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- We've derived the flexure formula for the case where the x-axis coincides with the beam axis (which is the neutral axis z=0)

- From this we see:
  - for $z<0$: $\sigma_x<0$ □ fibers above the neutral axis are in compression
  - for $z>0$: $\sigma_x>0$ □ fibers below the neutral axis are in tension

- For a beam in pure bending: normal stress is the only stress induced by the bending:

Normal Stress in Beams

- Due to the Poisson ratio we have strains in the y and z direction:

$$\varepsilon_y = \varepsilon_z = -\nu\varepsilon_x \qquad \varepsilon_x = \frac{\sigma_x(x,z)}{E} = \frac{M_y(x)\cdot z}{E\cdot I_y}$$

- In the case of a double symmetric beam we can write for the maximum normal stress (c is maximum distance away from neutral axis):

$$\sigma_{max} = \frac{M\cdot c}{I} = \frac{M}{S}$$

- S is the section module:

$$S = \frac{I}{c}$$

Comparing "Areas" for Tension, Torsion, and Bending

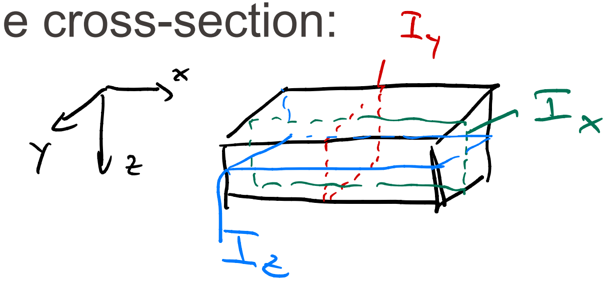
	Tension	Torsion	Bending
Stress formula	$\sigma = \frac{F}{A}$	$\tau(r) = \frac{T}{J} r$	$\sigma_x(z) = \frac{M}{I} z$
"Area" contribution	A	$J = \int_A r^2 dA$	$I = \int_A z^2 dA$

Finding Second Moment of Area and Centroid

- Second moment of area describes the resistance of a cross sectional area to bending around an axis in the plane of the cross-section:

$$I_x = \int_A y^2 dA$$

$$I_y = \int_A x^2 dA$$



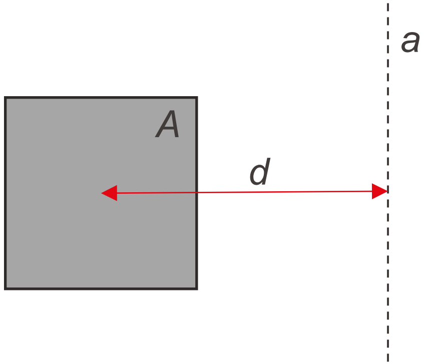
- This is not to be confused with the polar moment of inertia that describes the resistance of a cross section to torsion:

$$J_z = \int_A (x^2 + y^2) dA$$

- Perpendicular axis theorem:

$$J_z = I_x + I_y$$

Finding Second Moment of Area and Centroid

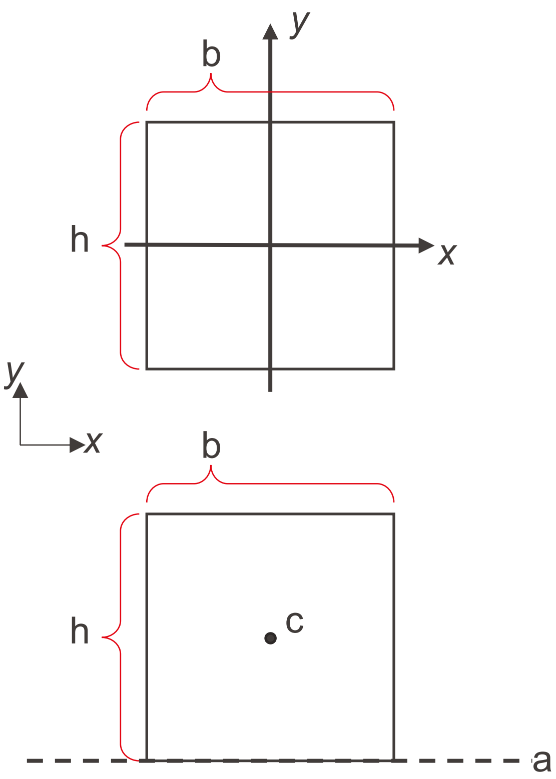


- The parallel axis theorem is used to calculate the second moment of area (I_a) around an axis *a*, when the second moment of area around an axis that passes through the center of gravity and is parallel to *a*, $(I_a)_{cg}$, is known. d is the distance from axis *a* to the center of gravity

$$I_a = (I_a)_{cg} + A \cdot d^2$$

- Centroid:
  - The geometrical center, the point from which gravity seems to act.
  - If an area has an axis of symmetry, the centroid is on that axis.

$$z_c = \frac{\int_A z dA}{\int_A dA} = \frac{\sum_i z_{ci} A_i}{\sum_i A_i}$$

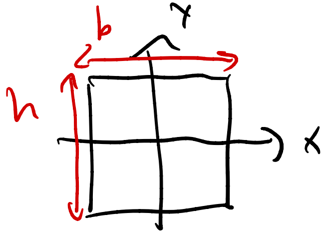


Example:
Second moment of area of a rectangle

- Calculate the second moment of area of the rectangle around the x-axis.

- Calculate the second moment of area around the bottom edge

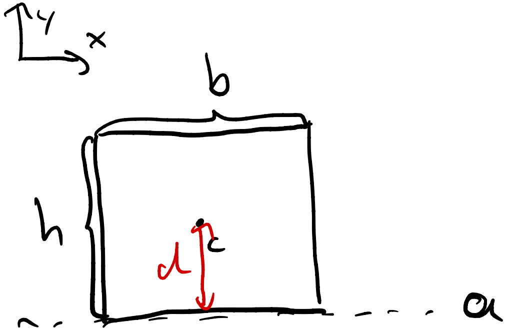
I_x:



$$I_x = \int_A y^2 \, dA = \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} y^2 \, dy \, dx$$

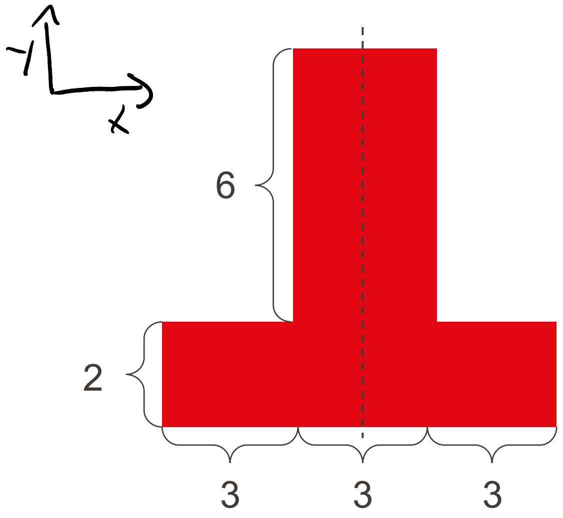
$$= \int_{-\frac{b}{2}}^{\frac{b}{2}} \left[\frac{y^3}{3} \Big|_{-\frac{h}{2}}^{\frac{h}{2}} \right] dx = \int_{-\frac{b}{2}}^{\frac{b}{2}} \left[\frac{1}{3} \frac{h^3}{8} + \frac{1}{3} \frac{h^3}{8} \right] dx = \int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{h^3}{12} \, dx$$

$$\underline{\underline{I_x}} = \frac{h^3}{12} \, x \Big|_{-\frac{b}{2}}^{\frac{b}{2}} = \underline{\underline{\frac{b \cdot h^3}{12}}}$$



$$I_{x,a} = I_x + \left(\frac{h}{2}\right)^2 \cdot A$$

$$= \frac{bh^3}{12} + \frac{h^2}{4} \cdot b \cdot h = \frac{bh^3}{3}$$



Georg Fantner

Example: Decomposition of complex cross-sections

Calculate for the beam cross-section:

- The centroid
- The second moment of area AROUND X

GIVEN: GEOMETRY

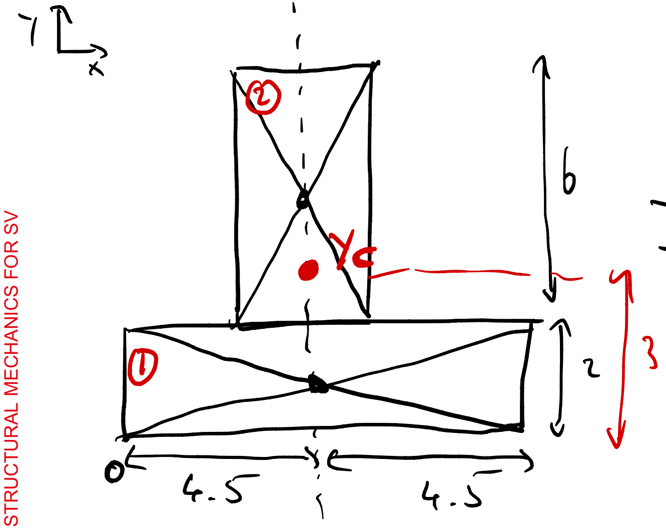
ASKED: x_c, y_c, I_x

APPROACH: DECOMPOSITION

APPLICABLE FORMULAS: $I_x = I_{x'} + Ad^2$

$$x_c = \frac{\sum x A}{\sum A}$$

$$I_x = \frac{b h^3}{12}$$



$x_c = 4.5$

BLOCK 1: $y_c^{①} = 1$

$A^{①} = 9 \cdot 2 = 18$

BLOCK 2: $y_c^{②} = 3 + 2 = 5$

$A^{②} = 3 \cdot 6 = 18$

Georg Fantner

BLOCK:	A_i	$y_{c,i}$	$y_{c,i} \cdot A$	I_i	$d_i = y_i - y_c$	$d_i^2 A_i$
①	18	1	18	$\frac{9 \cdot 2^3}{12} = 6$	-2	$4 \cdot 18 = 72$
②	18	5	90	$\frac{3 \cdot 6^3}{12} = 54$	2	$4 \cdot 18 = 72$

$$y_c = \frac{\sum A_i \cdot y_{c,i}}{\sum A_i} = \frac{18+90}{36} = 3$$

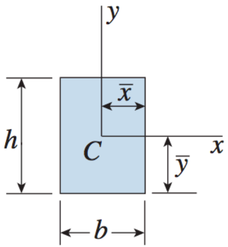
$$I_{x,c}^{①} = I^{①} + d_1^2 A_1 = 6 + 72 = 78$$

$$I_{x,c}^{②} = I^{②} + d_2^2 A_2 = 54 + 72 = 126$$

$$\underline{\underline{I_{x,c}}} = I_{x,c}^{①} + I_{x,c}^{②} = \underline{\underline{204}}$$

Some Standard 2nd Moments of Area

1

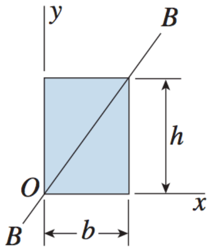


Rectangle (Origin of axes at centroid)

$$A = bh \qquad \bar{x} = \frac{b}{2} \qquad \bar{y} = \frac{h}{2}$$

$$I_x = \frac{bh^3}{12} \qquad I_y = \frac{hb^3}{12} \qquad I_{xy} = 0 \qquad I_P = \frac{bh}{12}(h^2 + b^2)$$

2

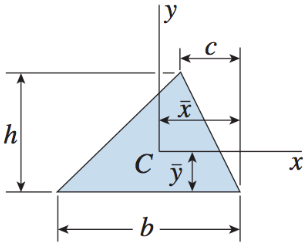


Rectangle (Origin of axes at corner)

$$I_x = \frac{bh^3}{3} \qquad I_y = \frac{hb^3}{3} \qquad I_{xy} = \frac{b^2h^2}{4} \qquad I_P = \frac{bh}{3}(h^2 + b^2)$$

$$I_{BB} = \frac{b^3h^3}{6(b^2 + h^2)}$$

3



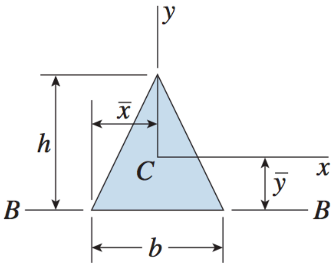
Triangle (Origin of axes at centroid)

$$A = \frac{bh}{2} \qquad \bar{x} = \frac{b + c}{3} \qquad \bar{y} = \frac{h}{3}$$

$$I_x = \frac{bh^3}{36} \qquad I_y = \frac{bh}{36}(b^2 - bc + c^2)$$

$$I_{xy} = \frac{bh^2}{72}(b - 2c) \qquad I_P = \frac{bh}{36}(h^2 + b^2 - bc + c^2)$$

5



Isosceles triangle (Origin of axes at centroid)

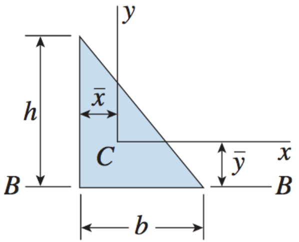
$$A = \frac{bh}{2} \qquad \bar{x} = \frac{b}{2} \qquad \bar{y} = \frac{h}{3}$$

$$I_x = \frac{bh^3}{36} \qquad I_y = \frac{hb^3}{48} \qquad I_{xy} = 0$$

$$I_P = \frac{bh}{144}(4h^2 + 3b^2) \qquad I_{BB} = \frac{bh^3}{12}$$

(*Note:* For an equilateral triangle, $h = \sqrt{3}\, b/2$.)

6



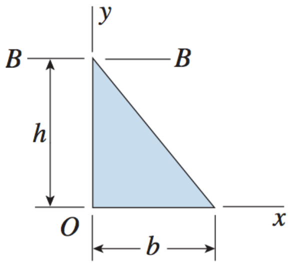
Right triangle (Origin of axes at centroid)

$$A = \frac{bh}{2} \qquad \bar{x} = \frac{b}{3} \qquad \bar{y} = \frac{h}{3}$$

$$I_x = \frac{bh^3}{36} \qquad I_y = \frac{hb^3}{36} \qquad I_{xy} = -\frac{b^2h^2}{72}$$

$$I_P = \frac{bh}{36}(h^2 + b^2) \qquad I_{BB} = \frac{bh^3}{12}$$

7

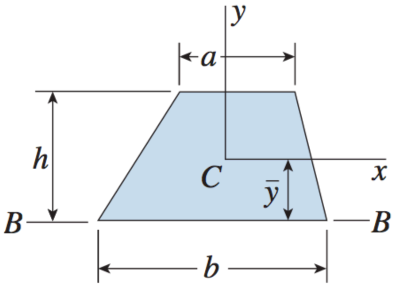


Right triangle (Origin of axes at vertex)

$$I_x = \frac{bh^3}{12} \qquad I_y = \frac{hb^3}{12} \qquad I_{xy} = \frac{b^2h^2}{24}$$

$$I_P = \frac{bh}{12}(h^2 + b^2) \qquad I_{BB} = \frac{bh^3}{4}$$

8

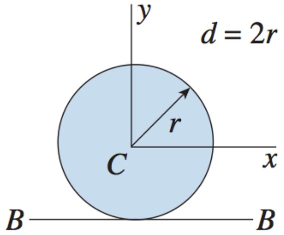


Trapezoid (Origin of axes at centroid)

$$A = \frac{h(a+b)}{2} \qquad \bar{y} = \frac{h(2a+b)}{3(a+b)}$$

$$I_x = \frac{h^3(a^2+4ab+b^2)}{36(a+b)} \qquad I_{BB} = \frac{h^3(3a+b)}{12}$$

9

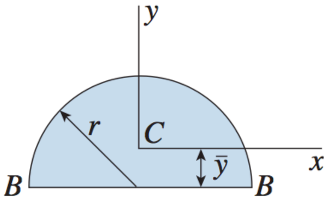


Circle (Origin of axes at center)

$$A = \pi r^2 = \frac{\pi d^2}{4} \qquad I_x = I_y = \frac{\pi r^4}{4} = \frac{\pi d^4}{64}$$

$$I_{xy} = 0 \qquad I_P = \frac{\pi r^4}{2} = \frac{\pi d^4}{32} \qquad I_{BB} = \frac{5\pi r^4}{4} = \frac{5\pi d^4}{64}$$

10



Semicircle (Origin of axes at centroid)

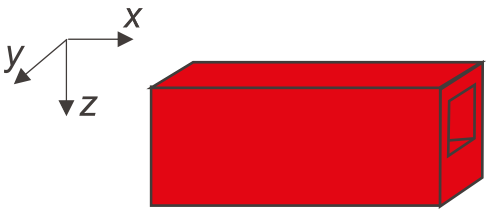
$$A = \frac{\pi r^2}{2} \qquad \bar{y} = \frac{4r}{3\pi}$$

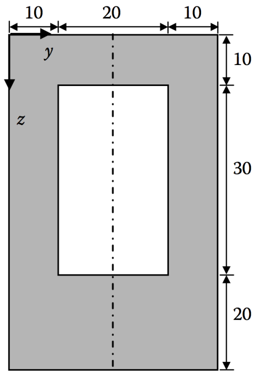
$$I_x = \frac{(9\pi^2 - 64)r^4}{72\pi} \approx 0.1098r^4 \qquad I_y = \frac{\pi r^4}{8} \qquad I_{xy} = 0 \qquad I_{BB} = \frac{\pi r^4}{8}$$

EPFL

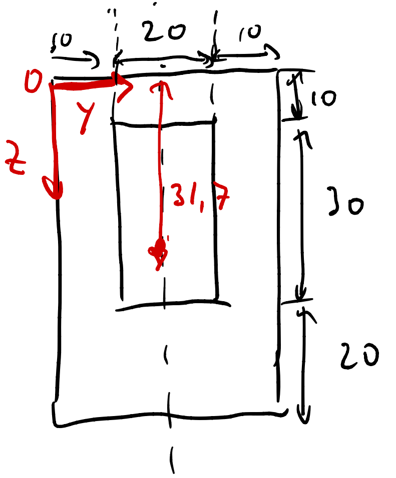
Example

Second moment of Area





Find the centroid and the second moment of area about the horizontal (*y*) axis of the cross section shown. All dimensions are in millimeters. If a beam is constructed with the cross section shown from steel whose maximum allowable tensile stress is 400 MPa, what is the maximum bending moment that may be applied to the beam?



given: - GEOMETRY,
- $\sigma_{max} = 400\ MPa$

ASKED: a z_c, y_c
b I_y
c M_{MAX}

APPROACH: SUPERPOSITION
+ PARAL. AXE THEOR.

$$I = \frac{bh^3}{12}$$

SOLUTION:

a CENTROID: SYMMETRY @ $y_c = 30$

$$z: \quad z_c = \frac{\sum z_{c,i} A_i}{\sum A_i}$$

BLOCKS	A_i	$z_{c,i}$	$z_{c,i} \cdot A_i$	I_i	$d_i = z_{c,i} - z_c$	$d_i^2 A_i$
①	$60 \cdot 40 = 2400$	30	72 000	$7.2 \cdot 10^5$	−1,67	6667
②	$20 \cdot 30 = 600$	25	15 000	$4.5 \cdot 10^4$	−6,67	26667

$$\Rightarrow z_c = \frac{\sum A_i z_{ci}}{\sum A_i} = \frac{57.000}{1800} = \frac{190}{6} \approx 31.7$$

Georg Fantner

Georg Fantner

b $$I_y = I_1 + d_1^2 A_1 - (I_2 + d_2^2 A_2)$$

$$= 7.2 \cdot 10^5 + 6667 - (4.5 \cdot 10^5 + 26'667)$$

$$= 6.55 \cdot 10^5 \, mm^4 = 6,55 \cdot 10^{-7} \, m^4$$

c $$\sigma_x = \frac{M \cdot z}{I}$$

$$\sigma_{MAX} = \sigma_x (z = c)$$

c is MAX DISTANCE FROM CENTROID

$c = 31.7$ mm

$$\sigma_{MAX} = \frac{M_{MAX} \cdot c}{I}$$

$$M_{MAX} = \frac{\sigma_{MAX} \, I}{c} = \frac{400 \cdot 10^6 \, N/m^2 \cdot 6.55 \cdot 10^{-7} \, m^4}{31.7 \cdot 10^{-3} \, m} = 8264 \, N \cdot m$$